

AUTOMATA, REDUCED WORDS AND GARSIDE SHADOWS IN COXETER GROUPS

CHRISTOPHE HOHLWEG[◊], PHILIPPE NADEAU, AND NATHAN WILLIAMS

ABSTRACT. In this article, we introduce and investigate a class of finite deterministic automata that all recognize the language of reduced words of a finitely generated Coxeter system (W, S) . The definition of these automata is straightforward as it only requires the notion of *weak order* on (W, S) and the related notion of *Garside shadows in* (W, S) , an analog of the notion of a Garside family. Then we discuss the relations between this class of automata and the canonical automaton built from Brink and Howlett's small roots. We end this article by providing partial positive answers to two conjectures: (1) the automata associated to the smallest Garside shadow is minimal; (2) the canonical automaton is minimal if and only if the support of all small roots is spherical, i.e., the corresponding root system is finite.

1. INTRODUCTION

In this article, we introduce and investigate a class of finite deterministic automata that recognize the language $\text{Red}(W, S)$ of reduced words of a finitely generated Coxeter system (W, S) . The definition of these automata is straightforward, requiring only the notion of *(right) weak order* \leq_R on (W, S) [1, 2] and the related notion of *Garside shadows*, introduced by M. Dyer and the first author in [13] as an analog of the notion of a Garside family in a monoid; see [11, 10] and the references therein. For general definitions and properties, we refer the reader to [25] regarding automata and to [2, 22] regarding Coxeter groups.

A *Garside shadow in* (W, S) is a subset $B \subseteq W$ that contains S and is closed under join (for the right weak order) and by taking suffixes. In [13], the authors show that finite Garside shadows exist in any Coxeter system (W, S) . Let B be a finite Garside shadow in (W, S) . So $\bigvee X \in B$ for any *bounded* subset X of B , i.e., a subset that has an upper bound. Therefore, the following projection from W to B is well-defined:

$$\begin{aligned} \pi_B : W &\rightarrow B \\ w &\mapsto \bigvee \{g \in B \mid g \leq_R w\} \end{aligned}$$

We denote by $\ell : W \rightarrow \mathbb{N}$ the *length function* of the Coxeter system (W, S) .

Definition 1.1. We define a finite deterministic automaton $\mathcal{A}_B(W, S)$ over the alphabet S as follows:

Date: April 18, 2016.

2010 Mathematics Subject Classification. Primary 20F55; secondary 20F10; 05E15; 06F99.

Key words and phrases. Coxeter groups, Garside shadows, low elements, weak order, dominance order, small roots, automaton and reduced words.

[◊]supported by NSERC Discovery grant *Coxeter groups and related structures*.

- the set of states is B ;
- the initial state is the identity e of W , and all states are final;
- the transitions are: $x \xrightarrow{s} \pi_B(sx)$ whenever $\ell(sx) > \ell(x)$.

Since the intersection of Garside shadows is again a Garside shadow, there is a smallest Garside shadow \tilde{S} in (W, S) . As a first example, the finite automaton built out of the smallest Garside shadow \tilde{S} for the infinite dihedral group is shown in Figure 1. Further examples are given in §3.6 and in Figures 5 and 6.

Our main result is that $\mathcal{A}_B(W, S)$ recognizes the language of reduced words of (W, S) .

Theorem 1.2. *If B is a finite Garside shadow in (W, S) , then the finite deterministic automaton $\mathcal{A}_B(W, S)$ recognizes the language $\text{Red}(W, S)$.*

Theorem 1.2 is proved in §2. In §3, we show that an inclusion $B \subseteq C$ of Garside shadows induces a surjective morphism $\mathcal{A}_C(W, S) \rightarrow \mathcal{A}_B(W, S)$ between their associated automata. The smallest Garside shadow being finite [13, Corollary 1.2], we are led to the following conjecture.

Conjecture 1. *The automaton $\mathcal{A}_{\tilde{S}}(W, S)$ is the minimal automaton recognizing $\text{Red}(W, S)$.*

Using Sage [23, S⁺09], we checked that Conjecture 1 holds for all Coxeter groups W of rank at most 4 whose corresponding Coxeter graph Γ_W has edge labels less than 10; see Remark 3.15 for more details.

Our initial motivation for this work was to provide a purely combinatorial definition for an automaton that recognizes the language of reduced words. Indeed, as we now recall, all previously-defined automata recognizing $\text{Red}(W, S)$ require the introduction of an auxiliary geometric representation and root system.

In 1993, B. Brink and R. Howlett [4] showed that finitely-generated Coxeter groups are automatic, in the sense of [16], thereby filling a gap in the proof of the “Parallel Wall Theorem” of [9]. For each Coxeter system (W, S) , they provided a *word-acceptor*—that is, a finite automaton that recognizes the language of lexicographically minimal reduced words in W . This particular automaton is built using their notion of *small roots*, and therefore requires a geometric representation of (W, S) and its associated root system. In a series of articles [7, 8, 5, 6], Casselman explains how to perform practical computations in Coxeter groups using Brink and Howlett’s word-acceptor.

We are often interested in *all* reduced words, not only those that are lexicographically-ordered; see for instance [26]. In his thesis [17], H. Eriksson studied a finite deterministic automaton $\mathcal{A}_0(W, S)$ over S that recognizes the language $\text{Red}(W, S)$. The automaton $\mathcal{A}_0(W, S)$ is called the *canonical automaton* in [2, §4.8], and is built using B. Brink and R. Howlett’s technology of small roots. An immediate consequence is that the language $\text{Red}(W, S)$ is regular, a result we recover in Theorem 1.2. In particular, the generating function for the number of reduced words in (W, S) with respect to their length is a rational function.

For $n \in \mathbb{N}$, the canonical automaton was extended, replacing small roots with n -small roots, in [15] and [13] to the n -canonical automaton $\mathcal{A}_n(W, S)$. We recall these notions in §3, and discuss morphisms between $\mathcal{A}_n(W, S)$ and the automata $\mathcal{A}_B(W, S)$ arising from certain finite Garside shadows B . In particular, we

show in Corollary 3.13 that any n -canonical automaton surjects into the automaton $\mathcal{A}_{\tilde{S}}(W, S)$, providing evidence for Conjecture 1.

Both H. Eriksson [17, Theorem 80] and P. Headley [19, Theorem V.8] prove that in type \tilde{A}_n , the canonical automaton $\mathcal{A}_0(\tilde{A}_n, S)$ is minimal. Furthermore, they note that $\mathcal{A}_0(W, S)$ is *not* minimal for general affine groups W .

We conjecture a necessary condition for the canonical automaton to be minimal. The sufficient condition is shown in Proposition 3.14.

Conjecture 2. *Let W be irreducible. Then $\mathcal{A}_0(W, S)$ is minimal if and only if $\Sigma = \Phi_{sph}^+$, where Φ_{sph}^+ denotes the set of roots whose support is a finite standard parabolic subgroup.*

Since $\mathcal{A}_0(W, S)$ surjects onto $\mathcal{A}_{\tilde{S}}(W, S)$ (Corollary 3.13), Conjecture 2 implies Conjecture 1 for Coxeter systems for which $\Sigma = \Phi_{sph}^+$. In §3.5, we prove Conjecture 2 in the following cases.

Theorem 1.3. *Conjecture 2 holds in each of the following cases:*

- (1) W is finite.
- (2) W is right-angled, i.e. $m_{st} = 2$ or ∞ for all $s \neq t$.
- (3) Γ_W is a complete graph, i.e. $m_{st} > 2$ for all $s \neq t$.
- (4) W is of type \tilde{A}_{n-1} .
- (5) W has rank 3.

In the first four cases, $\Sigma = \Phi_{sph}^+$ and $\mathcal{A}_0(W, S)$ is minimal.

We also checked that Conjecture 2 holds if W has rank 4 and $m_{st} < 10$ for all $s \neq t$; see Remark 3.15 for more details.

When (W, S) is an affine Coxeter system, P. Headley described a remarkable connection between the canonical automaton and the *Shi arrangement* [24]: the states of $\mathcal{A}_0(W, S)$ are in bijection with the (minimal elements in the) connected regions of the complement of the Shi arrangement for (W, S) [19]. The same relationship holds for the states of $\mathcal{A}_n(W, S)$ and the regions of the n -Shi arrangement, as we outline in §3.6.

2. GARSIDE SHADOW AUTOMATA

Fix (W, S) a Coxeter system with length function $\ell : W \rightarrow \mathbb{N}$. The *rank* of W is the cardinality of the set of *simple reflections* S . A word $s_1 \cdots s_k$ on the alphabet S is a *reduced word* for $w \in W$ if $w = s_1 \cdots s_k$ and $k = \ell(w)$. For $u, v, w \in W$, we say that:

- $w = uv$ is *reduced* if $\ell(w) = \ell(u) + \ell(v)$, i.e., the concatenation of any reduced word for u with any reduced word for v is a reduced word for w ;
- u is a *prefix* of w if a reduced word for u is a prefix of a reduced word for w ;
- v is a *suffix* of w if a reduced word for v is a suffix of a reduced word for w .

Observe that if $w = uv$ is reduced, then u is a prefix of w and v is a suffix of w . The subset $D_L(w) = \{s \in S \mid \ell(sw) < \ell(w)\}$ of S is called the *left descent set* of $w \in W$. The descent set plays an important role in the study of reduced words since it coincides with the set of the possible first letters of reduced words of an element $w \in W$; see [2].

The *standard parabolic subgroup* W_I is the subgroup of W generated by $I \subseteq S$. It is well-known that (W_I, I) is itself a Coxeter system and that the length function

$\ell_I : W_I \rightarrow \mathbb{N}$ is the restriction of ℓ to W_I . Moreover, W_I is finite if and only if it contains a *longest element*, which is then unique and is denoted by $w_{\circ, I}$.

The set $X_I := \{x \in W \mid \ell(sx) > \ell(x), \forall s \in I\}$ is the set of *minimal-length coset representatives for the coset $W_I \backslash W$* . For any $w \in W$, there is a unique decomposition $w = w_I w^I$, with $w_I w^I$ reduced; see [2, Proposition 2.4.4]. See [22, 2] for more details.

2.1. Weak order and Garside shadows. The (*right*) *weak order* is the order on W defined by $u \leq_R v$ if u is a prefix of v . Since we only consider the right weak order in this article, we only use from now on the term *weak order*. The weak order gives a natural orientation of the Cayley graph of (W, S) : for $w \in W$ and $s \in S$, we orient an edge $w \rightarrow ws$ if $w \leq_R ws$. We recall the following well-known useful properties linking descent sets and weak order, which is a rephrasing of part of [2, Proposition 3.1.2].

Lemma 2.1. *Let $u, v \in W$ and $s \in S$.*

- (a) *$s \in D_L(u)$ if and only if $s \leq_R u$.*
- (b) *If $s \in D_L(u) \cap D_L(v)$, then $u \leq_R v$ if and only if $su \leq_R sv$.*
- (c) *If $s \notin D_L(u)$ and $s \notin D_L(v)$, then $u \leq_R v$ if and only if $su \leq_R sv$.*

A. Björner [1, Theorem 8] proved that the weak order (W, \leq_R) is a complete meet semilattice: for any $A \subseteq W$, there exists an infimum $\bigwedge A \in W$, also called the *meet* of A ; see [2, Chapter 3].

A subset $X \subseteq W$ is *bounded in W* if there exists a $g \in W$ such that $x \leq_R g$ for any $x \in X$. Therefore, any bounded subset $X \subseteq W$ admits a least upper bound $\bigvee X$ called the *join* of X :

$$\bigvee X = \bigwedge \{g \in W \mid x \leq_R g, \forall x \in X\}.$$

When W is finite, any element $w \in W$ is a prefix of the longest element w_{\circ} , so that W itself is bounded. In fact, (W, \leq_R) turns out to be a complete ortholattice; see [2, Corollary 3.2.2].

Definition 2.2 ([13]). A subset $B \subseteq W$ is a *Garside shadow in (W, S)* if B contains S and:

- (i) B is closed under join in the weak order: if $X \subseteq B$ is bounded, then $\bigvee X \in B$;
- (ii) B is closed under taking suffixes: if $w \in B$, then any suffix of w is also in B .

Since a standard parabolic subgroup W_I with its canonical set of generators $I \subseteq S$ forms a Coxeter system, it is natural to say that a subset $B \subseteq W_I$ is a Garside shadow of (W_I, I) if B contains I and verifies Conditions (i)–(ii) of Definition 2.2. Note that if B is a Garside shadow in (W, S) , then $B \cap W_I$ is a Garside shadow in (W_I, I) [13, Remark 2.5(c)]. Since the intersection of Garside shadows is again a Garside shadow, there exists a smallest Garside shadow of (W, S) containing $X \subseteq W$, which we denote by $\text{Gar}_S(X)$. In [13, Corollary 1.2], Dyer and the first author show that the smallest Garside shadow

$$\tilde{S} := \text{Gar}_S(S)$$

is finite. The automaton constructed from the smallest Garside shadow \tilde{S} of the infinite dihedral group is illustrated in Figure 1.

Remark 2.3. The finiteness of \tilde{S} is shown in [13] using the geometry of the root system. A direct computational proof is still open. The problem of computing \tilde{S} relies on finding an efficient criterion for a subset of W to be bounded.

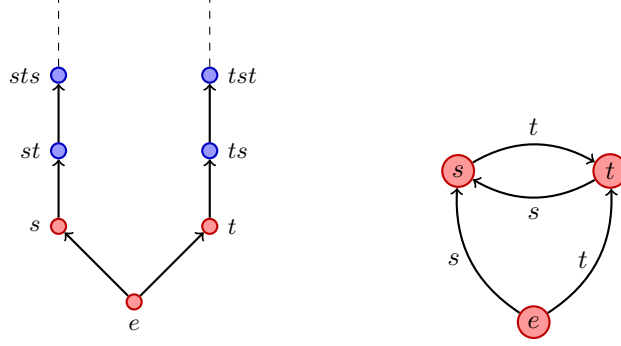


FIGURE 1. The weak order on the infinite dihedral group D_∞ and the automaton associated to the smallest Garside shadow $\{e, s, t\}$, which is represented by red vertices.

2.2. Garside Shadow Projections.

Definition 2.4. Let B be a Garside shadow in (W, S) . We call the surjection

$$\begin{aligned} \pi_B : W &\rightarrow B \\ w &\mapsto \bigvee \{g \in B \mid g \leq_R w\}, \end{aligned}$$

the B -projection.

Since $S \subseteq B$, the set $\{g \in B \mid g \leq_R w\}$ is non-empty and bounded for any $w \in W$. Together with Condition (i) of Definition 2.2, this implies that the B -projection is well-defined. Note that $\pi_B(w)$ can be characterized as the unique longest prefix of w which belongs to B .

Proposition 2.5. Let B be a Garside shadow in (W, S) and $u, w \in W$, then:

- (a) $\pi_B \circ \pi_B = \pi_B$;
- (b) $\pi_B(w) \leq_R w$, with equality holding if and only if $w \in B$;
- (c) If $u \leq_R w$, then $\pi_B(u) \leq_R \pi_B(w)$.

Proof. Properties (a) and (b) are clear from the definition. For (c), if $u \leq_R w$ then $x \leq_R u$ implies $x \leq_R w$ for any $x \in B$, from which we conclude the proposition. \square

The next proposition states that left descent sets are invariant under Garside shadow projections.

Proposition 2.6. Let B be a Garside shadow in (W, S) and $w \in W$. Then $D_L(w) = D_L(\pi_B(w))$, and $s\pi_B(w) \leq_R sw$ for any $s \in S$.

Proof. We first show $D_L(w) = D_L(\pi_B(w))$. Let $r \in D_L(\pi_B(w))$. By Lemma 2.1(a) and Proposition 2.5(b), we have $r \leq_R \pi_B(w) \leq_R w$. So $D_L(\pi_B(w)) \subseteq D_L(w)$.

Conversely, let $s \leq_R w$. Since $S \subseteq B$, we have $s = \pi_B(s)$ by Proposition 2.5(b). Then $s \leq_R \pi_B(w)$ by Proposition 2.5(c). So $D_L(w) \subseteq D_L(\pi_B(w))$.

We now show $s\pi_B(w) \leq_R sw$ for any $s \in S$. We write $D := D_L(w) = D_L(\pi_B(w))$. If $s \notin D$, then the result follows from Proposition 2.5(b) and Lemma 2.1(c). If $s \in D$, the statement follows from Proposition 2.5(b) and Lemma 2.1(b). \square

Remark 2.7. Definition 2.4 of π_B and the proof of Proposition 2.6 require only the conditions $S \subseteq B$ and Condition (i) from Definition 2.2. The proof of Proposition 2.5 only uses properties of the weak order.

The next result is crucial for proving Theorem 1.2.

Proposition 2.8. *Let B be a Garside shadow in (W, S) . Let $w \in W$ and $s \in S$ such that $s \notin D_L(w)$. Then $\pi_B(sw) = \pi_B(s\pi_B(w))$.*

Proof. We have $s\pi_B(w) \leq sw$ by Proposition 2.6. Therefore, by Proposition 2.5(c), $\pi_B(s\pi_B(w)) \leq_R \pi_B(sw)$. To complete the proof, we will show that $\pi_B(sw) \leq_R \pi_B(s\pi_B(w))$, which is equivalent by Proposition 2.5 (a), (b), and (c) to the statement that $\pi_B(sw) \leq_R s\pi_B(w)$. We prove this last relation as follows. Using Proposition 2.6, we see that $s \in D_L(sw) = D_L(\pi_B(sw))$, so that since $\pi_B(sw) \leq_R sw$ by Proposition 2.5(b), Lemma 2.1(b) allows us to conclude that $s\pi_B(sw) \leq_R w$. Now $s\pi_B(sw) \in B$ because B is closed under taking suffixes by Definition 2.2(ii), so that $\pi_B(s\pi_B(sw)) = s\pi_B(sw) \leq_R \pi_B(w)$ by Proposition 2.5(b,c). Multiplying both sides by s and using Lemma 2.1(c) gives $\pi_B(sw) \leq_R s\pi_B(w)$. \square

Remark 2.9. The proof of Proposition 2.8 requires all the conditions from Definition 2.2.

Corollary 2.10. *Let $v \in W$ and $s_1, \dots, s_k \in S$ such that $s_k \cdots s_1 v$ is reduced. Then*

$$\pi_B(s_k \cdots s_1 v) = \pi_B(s_k \pi_B(s_{k-1} \pi_B(\cdots s_2 \pi_B(s_1 \pi_B(v))))) = \pi_B(s_k \cdots s_1 \pi_B(v)).$$

In particular:

(a) *If $w = s_k \cdots s_1$ is a reduced word, then*

$$\pi_B(w) = \pi_B(s_k \cdots s_1) = \pi_B(s_k \pi_B(s_{k-1} \pi_B(\cdots s_2 \pi_B(s_1))));$$

(b) *If uv is reduced, $u \in W$, then $\pi_B(u\pi_B(v)) = \pi_B(uv)$.*

Proof. We prove the first equality by induction on $k > 0$. The case $k = 1$ is Proposition 2.8. Now assume the property for $k - 1 > 0$. Then since $s_k s_{k-1} \cdots s_1 v$ is reduced, we have $s_k \notin D_L(s_{k-1} \cdots s_1 v)$. By Proposition 2.8, we obtain

$$\pi_B(s_k \cdots s_1 v) = \pi_B(s_k \pi_B(s_{k-1} \cdots s_1 v)).$$

By induction, $\pi_B(s_{k-1} \cdots s_1 v) = \pi_B(s_{k-1} \pi_B(\cdots s_2 \pi_B(s_1 \pi_B(v))))$.

Now for the second equality, observe that $\pi_B(v)$ is a prefix of v by Proposition 2.5(b). Therefore $s_k \cdots s_1 \pi_B(v)$ must be reduced, since $s_k \cdots s_1 v$ is reduced. We conclude by applying the first equality to $\pi_B(v)$, recalling that π_B is a projection. In particular (a) is obtained by taking $v = e$ and (b) by considering a reduced word $s_k \cdots s_1$ for u . \square

2.3. Garside Shadow Automata and Proof of Theorem 1.2. Before proving Theorem 1.2, we recall some terminology about automata theory; see [25]. A *finite deterministic automaton* \mathcal{A} over the alphabet S is a quadruple (Q, q_0, F, δ) where Q is a finite set of *states*, $q_0 \in Q$ is the *initial state*, $F \subseteq Q$ is the set of *final states*, and δ is a partial function $Q \times S \rightarrow Q$. If $\delta(q, s) = q'$ then $q \xrightarrow{s} q'$ is a *transition*. An automaton \mathcal{A} can thus be seen as a directed graph on the vertex set Q with edges labeled by elements of S such that for any q, s there is at most one edge with source q and label s .

For an automaton \mathcal{A} , one naturally extends δ to a partial function $Q \times S^* \rightarrow Q$. A word $s_1 \cdots s_k \in S^*$ is *accepted* by \mathcal{A} if $\delta(q_0, s_1 \cdots s_k)$ is defined and is in F . The set of all words accepted by \mathcal{A} is the *language recognized* by \mathcal{A} and denoted by $\mathcal{L}(\mathcal{A})$. Languages $L \subseteq S^*$ occurring in this way are called *regular*, and it is a fundamental theorem of Kleene that the class of such languages coincides with the class of *rational languages*; see [25, Theorem 2.1].

Let B be a Garside shadow. Recall from Definition 1.1 in the introduction that the automaton $\mathcal{A}_B(W, S)$ is defined by:

- the set of states is B ;
- the initial state is the identity e of W , and all states are final;
- the transitions are: $x \xrightarrow{s} \pi_B(sx)$ whenever $s \notin D_L(x)$.

We denote $\mathcal{A}_B := \mathcal{A}_B(W, S)$ if there is no possible confusion. We prove now that $\mathcal{A}_B(W, S)$ recognizes the language $\text{Red}(W, S)$ of reduced words in (W, S) .

Proof of Theorem 1.2. We prove the theorem by induction. Let $\mathcal{P}(k)$ ($k \in \mathbb{N}$) be the following property:

For any sequence s_1, \dots, s_k of simple reflections, $s_k \cdots s_1$ is reduced if and only if there is a path in \mathcal{A}_B starting at the initial state e with edges labeled successively by s_1, \dots, s_k . The final state of such a path is $\pi_B(s_k \cdots s_1)$.

By definition of \mathcal{A}_B , properties $\mathcal{P}(0)$ and $\mathcal{P}(1)$ are easily seen to be true. Now let $k > 1$ be such that $\mathcal{P}(i)$ holds for all $i < k$, and consider any sequence $s_1, \dots, s_k \in S$. Let $w_j := s_j \cdots s_1$, and let $x_j := \pi_B(s_j x_{j-1})$.

We first show that the sequence of edge labels for a path in \mathcal{A}_B is reduced. Suppose there is a path in \mathcal{A}_B

$$e \xrightarrow{s_1} x_1 \xrightarrow{s_2} x_2 \xrightarrow{s_3} \cdots \xrightarrow{s_{k-1}} x_{k-1} \xrightarrow{s_k} x_k$$

from the state e to the state x_k , with edges labeled by s_1, \dots, s_k . By induction, $s_{k-1} \cdots s_1$ is reduced, so that by Corollary 2.10 $\pi_B(w_{k-1}) = x_{k-1}$. Since $x_{k-1} \xrightarrow{s_k} x_k$ is an edge in the automaton \mathcal{A}_B , $s_k \notin D_L(x_{k-1})$ by definition. Therefore $s_k \notin D_L(w_{k-1})$, since $D_L(x_{k-1}) = D_L(w_{k-1})$ by Proposition 2.6. In particular, since $s_{k-1} \cdots s_1$ is reduced, $s_k w_{k-1} = s_k s_{k-1} \cdots s_1$ is also reduced.

We now show that any reduced word $s_k s_{k-1} \cdots s_1$ gives a path in \mathcal{A}_B from e , with the desired edge labels and ending state. For the sake of contradiction, suppose that the sequence s_1, \dots, s_k does not define a path in \mathcal{A}_B .

This gives rise to two cases, both of which lead to a contradiction of our initial assumption that $s_k s_{k-1} \cdots s_1$ is reduced. If the initial sequence s_1, \dots, s_{k-1} does not define a path in \mathcal{A}_B , then, by induction $s_{k-1} \cdots s_1$ is not reduced, contradicting our assumption. Otherwise, the sequence s_1, \dots, s_{k-1} ends at the state x_{k-1} and, by induction, $s_{k-1} \cdots s_1$ is reduced. In particular, $\pi_B(w_{k-1}) = x_{k-1}$ by Corollary 2.10. Since s_1, \dots, s_k does not define a path, $s_k \in D_L(x_{k-1})$, so that $s_k \in D_L(w_{k-1})$ by

Proposition 2.6. But then $s_k s_{k-1} \cdots s_1 = s_k w_{k-1}$ is not reduced, which again contradicts our initial assumption. \square

Remark 2.11. Neither the definition of $\mathcal{A}_B(W, S)$, the definition of π_B nor the proof of Theorem 1.2 requires B to be finite. However we chose to state the result for finite Garside shadows in Theorem 1.2, since those produce finite automata.

2.4. Root systems and inversion sets. Before studying the relation between Garside shadow automata and standard parabolic subgroups, we need to introduce a geometric representation and a root system for (W, S) .

Recall that a quadratic space (V, B) is a data of a real vector space V with a symmetric bilinear form B . The group $O_B(V)$ is the group consisting of all linear maps that preserve B . For any non-isotropic vector $\alpha \in V$, i.e., $B(\alpha, \alpha) \neq 0$, we associate a B -reflection s_α given by the formula $s_\alpha(v) = v - 2 \frac{B(\alpha, v)}{B(\alpha, \alpha)} \alpha$, for all $v \in V$.

We consider now a *geometric representation of (W, S)* , i.e., a faithful representation of W as a subgroup of $O_B(V)$, where S is mapped into the set of B -reflections associated to a *simple system* $\Delta = \{\alpha_s \mid s \in S\}$ ($s = s_{\alpha_s}$). Then the W -orbit $\Phi = W(\Delta)$ is a *root system* with *positive roots* $\Phi^+ = \text{cone}_\Phi(\Delta)$ and *negative roots* $\Phi^- = -\Phi^+$, where $\text{cone}(X)$ is the set of nonnegative linear combination of vectors in $X \subseteq V$ and $\text{cone}_\Phi(X) = \text{cone}(X) \cap \Phi$; see [21, §1] for more details.

We recall now some useful well-known results linking roots and reduced words in (W, S) .

The *left inversion set* of $w \in W$ of $w \in W$ is defined by $N(w) := \Phi^+ \cap w(\Phi^-)$. The following proposition may be found in [20, §2.3-§2.5]; part (b) is due to M Dyer [12].

Proposition 2.12. *The map $N : (W, \leq_R) \rightarrow (\mathcal{P}(\Phi^+), \subseteq)$ is a poset monomorphism. Furthermore:*

- (a) *For any $u, w \in W$, $u \leq_R w$ if and only if $N(u) \subseteq N(w)$;*
- (b) *For any bounded $X \subseteq W$, $N(\bigvee X) = \text{cone}_\Phi(\bigcup_{x \in X} N(x))$.*

If $I \subseteq S$, then $\Delta_I := \{\alpha_s \mid s \in I\}$ is a simple system with root system $\Phi_I := W_I(\Delta_I)$ and positive root system $\Phi_I^+ := \Phi_I \cap \Phi^+$ for the standard parabolic subgroup W_I . The following statement is well-known; we include a proof here for completeness.

Corollary 2.13. *Let $I \subseteq S$ and $w = w_I w^I$ with $w_I \in W_I$ and $w^I \in X_I$, then $N(w_I) = N(w) \cap \Phi_I^+$.*

Proof. The left-to-right inclusion follows from Proposition 2.12(a) since w_I is a prefix of w . Now let $\alpha \in N(w) \cap \Phi_I^+$. So $w_I^{-1}(\alpha) \in \Phi_I$ and $w^{-1}(\alpha) = (w^I)^{-1} w_I^{-1}(\alpha)$ is an element of Φ^- . Now $(w^I)^{-1}(\Phi_I^+) \subseteq \Phi^+$, since $w^I \in X_I$. Thus $w_I^{-1}(\alpha)$ must be in Φ_I^- , and so $\alpha \in N(w_I)$. \square

2.5. Parabolic Subgroups. We now discuss the behaviour of Garside shadows with respect to standard parabolic subgroups.

Let B be a Garside shadow in (W, S) and W_I be the standard parabolic subgroup generated by $I \subseteq S$. Then $B \cap W_I$ is a Garside shadow [13, Remark 2.5(c)]. Let $\mathcal{A}_B^{(I)}(W, S)$ be the restriction of the automaton $\mathcal{A}_B(W, S)$ to the states corresponding to $B \cap W_I$ and to the transitions corresponding to $s \in I$.

Proposition 2.14. *Let B be a Garside shadow and $I \subseteq S$.*

- (a) The restriction of π_B to W_I is the $(B \cap W_I)$ -projection $\pi_{B \cap W_I} : W_I \rightarrow B \cap W_I$.
(b) $\mathcal{A}_B^{(I)}(W, S) = \mathcal{A}_{B \cap W_I}(W_I, I)$.

Proof. (a) By definition of Garside shadow projections, we need to show that for any $w \in W_I$ we have $\{g \in B \mid g \leq_R w\} = \{g \in B \cap W_I \mid g \leq_R w\}$.

The right-to-left inclusion is obvious. Now let $w \in W_I$ and $g \leq_R w$. Since $w \in W_I$, any reduced word for w uses only letters from I , by [2, Corollary 1.4.8(ii)]. In particular any prefix of w is in W_I . Since $g \leq_R w$, g is a prefix of W_I . Therefore $g \in W_I$, which concludes the proof of (a).

(b) By definition, the states of $\mathcal{A}_B^{(I)}(W, S)$ and of $\mathcal{A}_{B \cap W_I}(W_I, I)$ are the same. The fact that the transitions are the same follows by (a). \square

Remark 2.15 (Minimal automata and restriction to standard parabolic subgroups). We do not know if the restriction of a Garside shadow to W_I is compatible with Garside closure [13, Remark 2.5(c)]. In other words, if B is a Garside shadow in (W, S) , is $\text{Gar}_S(B) \cap W_I = B$? In particular, we do not know if $\mathcal{A}_S^{(I)}(W, S) = \mathcal{A}_I(W_I, I)$.

Another way to restrict a Garside shadow to a standard parabolic subgroup is by the mean of the minimal coset representatives decomposition; the associated automaton structure is discussed in Proposition 3.2. Recall that any element $w \in W$ has a unique decomposition $w = w_I w^I$ with $w_I \in W_I$ and $w^I \in X_I$. We denote by $p_I : W \rightarrow W_I$ the projection defined by $p_I(w) := w_I$.

Proposition 2.16. *Let B be a Garside shadow in (W, S) and $I \subseteq S$.*

- (a) *The set $p_I(B)$ is a Garside shadow in (W_I, I) .*
(b) *We have $p_I \circ \pi_B = \pi_{p_I(B)} \circ p_I$.*

Remark 2.17. We have $B \cap W_I \subseteq p_I(B)$, but equality does not hold in general. Indeed, let $S = \{s, t, u\}$ and $W = \{s \mid s^2 = t^2 = u^2 = 1, su = us\}$. One checks that $B := \{1, s, t, u, su, tu, stu\}$ is a Garside shadow for (W, S) . Now pick $I = \{s, t\}$. Then we have $p_I(stu) = st \notin B$ while $stu \in B$, so $st \in p_I(B) \setminus (B \cap W_I)$.

To prove the proposition, we need the following lemma.

Lemma 2.18. *Let X be a bounded set in W and $I \subseteq S$, then $p_I(\bigvee X) = \bigvee p_I(X)$.*

Proof. By Corollary 2.13 and Proposition 2.12(b) we have

$$N\left(p_I\left(\bigvee X\right)\right) = N\left(\bigvee X\right) \cap \Phi_I = \text{cone}_\Phi\left(\bigcup_{x \in X} N(x)\right) \cap \Phi_I.$$

Since our statement is about combinatorics of reduced words, we consider without loss of generality the simple system to be a basis of V . So in particular $\text{span}(\Phi_I)$ is a supporting hyperplane of $\text{cone}(\Delta)$. Therefore,

$$\text{cone}_\Phi\left(\bigcup_{x \in X} N(x)\right) \cap \Phi_I = \text{cone}_\Phi\left(\bigcup_{x \in X} N(x) \cap \Phi_I\right),$$

since there are only finitely many generators for each cone. So by Corollary 2.13 we obtain

$$N\left(p_I\left(\bigvee X\right)\right) = \text{cone}_{\Phi_I}\left(\bigcup_{x \in X} N(p_I(x))\right).$$

Finally, by Proposition 2.12(b) and Corollary 2.13 again we have:

$$N\left(\bigvee p_I(X)\right) = \text{cone}_{\Phi_I}\left(\bigcup_{x \in X} N(p_I(x))\right) = N\left(p_I\left(\bigvee X\right)\right). \quad \square$$

Proof of Proposition 2.16. (a) We verify the conditions in Definition 2.2. It is clear that $p_I(B) \subseteq W_I$. Now, since $p_I(s) = s$ for any $s \in I$ and $I \subseteq S \subseteq B$ we have $I \subseteq p_I(B)$. For Condition (i), consider $X \subseteq B$ bounded in W . So $p_I(X)$ is bounded in W , so its join $\bigvee p_I(X)$ exists. We have to show that $\bigvee p_I(X) \in p_I(B)$. By Lemma 2.18 we have $\bigvee p_I(X) = p_I(\bigvee X)$, which is an element of $p_I(B)$ since $\bigvee X \in B$. For Condition (ii), consider $w \in B$ and a suffix v of w_I . Since $w^I \in X_I$, the expression vw_I is reduced. Therefore, vw^I is a suffix of $w \in B$. Since B is a Garside shadow, $vw^I \in B$. Furthermore, $p_I(vw^I) = v$, so $v \in p_I(B)$.

(b) It is enough to show that for $w \in W$, we have

$$\{p_I(g) \mid g \in B, g \leq_R w\} = \{g' \in p_I(B) \mid g' \leq_R p_I(w)\}.$$

But this follows easily from Proposition 2.12(a) together with Corollary 2.13. \square

3. MORPHISMS AND GARSIDE SHADOW AUTOMATA

In this section, we discuss morphisms between Garside shadow automata, then we compare the automata of a particular family of Garside shadows, the set of n -low elements with the family of n -canonical automata. We first recall the definitions of morphisms of automata, minimal automata, and the concept of minimal roots for (W, S) .

3.1. Morphisms of automata. We refer the reader to [25, Chapter II(3)] for additional details on morphisms of automata. Here we shall only use this notion in a particular case suited to our various automata.

Definition 3.1 (see [25, Chapter II(3)]). Let $\mathcal{A} = (Q, q_0, F, \delta)$ and $\mathcal{A}' = (Q', q'_0, F', \delta')$ be two finite deterministic automata over the same alphabet S . A function $f : Q \rightarrow Q'$ is a *morphism of automata* between \mathcal{A} and \mathcal{A}' if

- (i) $f(q_0) = q'_0$;
- (ii) $f(F) \subseteq F'$;
- (iii) If $q_1 \xrightarrow{s} q_2$ is a transition in \mathcal{A} , then $f(q_1) \xrightarrow{s} f(q_2)$ is a transition in \mathcal{A}' .

A morphism of automata f is *totally surjective* if f is surjective, satisfies $f^{-1}(F') = F$ and if, for any transition $q'_1 \xrightarrow{s} q'_2$ in \mathcal{A}' , there exists q_1, q_2 such that $f(q_1) = q'_1$, $f(q_2) = q'_2$ and $q_1 \xrightarrow{s} q_2$ in \mathcal{A} . In this case \mathcal{A}' is called a *quotient* of \mathcal{A} .

If f is a morphism between \mathcal{A} and \mathcal{A}' then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$. If f is totally surjective then $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathcal{A}')$.

The following proposition gives a first example of a totally surjective morphism related to Garside shadow automata and arising from the surjection p_I from Proposition 2.16.

Proposition 3.2. *Let $I \subseteq S$ and B be a Garside shadow in (W, S) . The automaton $\mathcal{A}_B(W, I)$ is defined by taking the same states B , initial state e , and final states as $\mathcal{A}_B(W, S)$, but with only the transitions of $\mathcal{A}_B(W, S)$ corresponding to letters in I . Then the surjection $p_I : B \rightarrow p_I(B)$ induces a totally surjective morphism from $\mathcal{A}_B(W, I)$ to $\mathcal{A}_{p_I(B)}(W_I, I)$.*

Proof. Let us first show that p_I verifies the conditions in Definition 3.1. We have $p_I(e) = e$, and all states are final in both $\mathcal{A}_B(W, I)$ and $\mathcal{A}_{p_I(B)}(W_I, I)$, so (i) and (ii) hold. To prove (iii), let $w \xrightarrow{s} \pi_B(sw)$ be a transition in $\mathcal{A}_B(W, I)$ with $s \in I \setminus D_L(w)$. We have to show that $p_I(w) \xrightarrow{s} p_I(\pi_B(sw))$ is a transition in $\mathcal{A}_{p_I(B)}(W_I, I)$. Since $D_L(w) \cap I = D_L(p_I(w))$, there is a transition $p_I(w) \xrightarrow{s} \pi_{p_I(B)}(sp_I(w))$. The equality $p_I(\pi_B(sw)) = \pi_{p_I(B)}(sp_I(w))$ is then guaranteed by Proposition 2.16(b) since $p_I(sw) = sp_I(w)$ for any $s \in I \setminus D_L(w)$.

To prove that p_I is totally surjective, let $p_I(w) \xrightarrow{s} \pi_{p_I(B)}(sp_I(w))$ be a transition in $\mathcal{A}_{p_I(B)}(W_I, I)$, with $w \in B$ and $s \in I \setminus D_L(p_I(w))$. Then $w \xrightarrow{s} \pi_B(sw)$ is a transition in $\mathcal{A}_B(W, I)$ since $D_L(w) \cap I = D_L(p_I(w))$, and we conclude as in the previous paragraph. \square

Minimal automata. Given a regular language $L \in S^*$, there exists an automaton $\mathcal{R}(L)$ which recognizes L and is a quotient of all automata that recognize L , called the *minimal automaton of L* .

It can be constructed as follows: given $u \in S^*$, define $u^{-1}L$ to be the set of $v \in S^*$ such that $uv \in L$, and let $Q_L = \{u^{-1}L \mid u \in S^*\}$. We then define $\mathcal{R}(L) = (Q_L, q_L, F_L, \delta_L)$ with $q_L = L$, $F_L = \{u^{-1}L \mid u \in L\}$ and transitions $\delta_L(u^{-1}L, a) = (ua)^{-1}L$. This automaton clearly recognizes L .

Now pick any deterministic, complete¹ automaton \mathcal{A} such that $L = \mathcal{L}(\mathcal{A})$. Given $q \in Q$, let $L_q(\mathcal{A})$ be the language recognized by the automaton \mathcal{A} with q replacing q_0 as initial state. If $L_q(\mathcal{A}) = L_{q'}(\mathcal{A})$ then q and q' are called *equivalent* states. Then $q \mapsto L_q(\mathcal{A})$ is a totally surjective morphism from \mathcal{A} to $\mathcal{R}(L)$. Therefore in order to prove that an automaton is minimal, one must show that distinct states are never equivalent.

Remark 3.3. Denote by $\mathcal{A}_{\min}(W, S)$ the minimal automaton that recognizes the language $\text{Red}(W, S)$. For $I \subseteq S$, we define the automaton $\mathcal{A}_{\min}^{(I)}(W, S)$ to be the restriction of $\mathcal{A}_{\min}(W, S)$ to the transitions in I and the states that can be reached from the initial state using these transitions. We now show that $\mathcal{A}_{\min}^{(I)}(W, S) = \mathcal{A}_{\min}(W_I, I)$, so that minimal automata remain minimal upon restriction to a parabolic subgroup.

Let q_1, q_2 be distinct states in $\mathcal{A}_{\min}^{(I)}(W, S)$; q_1, q_2 can be reached by reading (reduced words for elements) $w_1, w_2 \in W_I$, respectively. Since q_1, q_2 are non-equivalent states in $\mathcal{A}_{\min}(W, S)$ by minimality, there exists $w \in W$ such that w_1w is reduced while w_2w is not reduced. Now use the decomposition $w = w_I w^I$ with $w_I \in W_I$ and $w^I \in X_I$, then w_1w_I is reduced while w_2w_I is not reduced. So the states are not equivalent in $\mathcal{A}_{\min}^{(I)}(W, S)$, which is therefore the minimal automaton $\mathcal{A}_{\min}(W_I, I)$ that recognizes $\text{Red}(W_I, I)$.

Therefore—assuming Conjecture 1—we conclude that $\mathcal{A}_S^{(I)}(W, S) = \mathcal{A}_I(W_I, I)$, resolving our question in Remark 2.15.

3.2. Inclusion of Garside shadows and morphisms of automata.

Proposition 3.4. *If $C \subseteq B$ are two Garside shadows, then $\pi_C \circ \pi_B = \pi_C$.*

¹This means that δ is defined everywhere. Any automaton can be transformed into a complete one by adding a non-final *sink state* \dagger and transitions $\delta(q, s) := \dagger$ whenever δ is not previously defined.

Proof. It is enough to show that for $w \in W$, we have

$$\{g \in C \mid g \leq_R \pi_B(w)\} = \{g \in C \mid g \leq_R w\}.$$

The left-to-right inclusion follows from Proposition 2.5(b). Now let $g \in C$ such that $g \leq_R w$. Since $C \subseteq B$ we have by Proposition 2.5(b,c) that

$$g = \pi_B(g) \leq_R \pi_B(w),$$

which concludes the proof. \square

Corollary 3.5. *If $C \subseteq B$ are two Garside shadows, then the C -projection π_C induces a totally surjective morphism from \mathcal{A}_B to \mathcal{A}_C . In particular, $\mathcal{A}_{\tilde{S}}$ is a quotient of any Garside shadow automaton.*

Proof. The C -projection $\pi_C : B \rightarrow C$ is surjective, since if $w \in C$, then $w \in B$ and $\pi_C(w) = w$. We now show that π_C verifies the conditions in Definition 3.1.

(i) Since $e \in C$, $\pi_C(e) = e$.

(ii) Let $w \xrightarrow{s} \pi_B(sw)$ be a transition in \mathcal{A}_B with $s \notin D_L(w)$. We have to show that $\pi_C(w) \xrightarrow{s} \pi_C(\pi_B(sw)) = \pi_C(sw)$ is a transition in \mathcal{A}_C , using Proposition 3.4. Since $D_L(w) = D_L(\pi_C(w))$ by Proposition 2.6, we have $s \notin D_L(\pi_C(w))$. So $\pi_C(w) \xrightarrow{s} \pi_C(s\pi_C(w))$ is a transition in \mathcal{A}_C by definition, and we conclude by Proposition 2.8.

(iii) This holds since all states are final in both automata and π_C is surjective.

To prove that π_C is totally surjective, it remains to show that, if $v \xrightarrow{s} v'$ is a transition in \mathcal{A}_C , there is a transition $u \xrightarrow{s} u'$ in \mathcal{A}_B with $\pi_C(u) = v$ and $\pi_C(u') = v'$. But this is guaranteed by taking $u := v$ and $u' := v$ since $C \subseteq B$. \square

To conclude this discussion, we show Conjecture 1 in the finite case.

Proposition 3.6. *Assume that W is finite. Then $\tilde{S} = W$ and $\mathcal{A}_{\tilde{S}}$ is the minimal automaton that recognizes $\text{Red}(W, S)$.*

Proof. The fact that W is finite implies that $\tilde{S} = W$ [13, Proposition 2.2(3)]. Therefore $\pi_{\tilde{S}} = \pi_W = \text{Id}_W$, the identity map on W . Thus $\mathcal{A}_{\tilde{S}}$ has states indexed by W and transitions $w \xrightarrow{s} sw$ if $s \notin D_L(w)$.

Let $u, v \in W$ be two equivalent states, i.e. for any s_1, s_2, \dots, s_k , we have that $s_k \cdots s_1 u$ is reduced if and only if $s_k \cdots s_1 v$ is reduced. We must prove that $u = v$. Let $k \geq 0$ be maximal so that there exist s_1, s_2, \dots, s_k with $s_k \cdots s_1 u$ reduced; note that k exists since W is finite. By hypothesis, $s_k \cdots s_1 v$ is reduced and k is necessarily also maximal for that property. But there is a unique element w satisfying $D_L(w) = S$, namely the longest element w_\circ [2, Proposition 2.3.1(ii)]. This shows that $s_k \cdots s_1 u = w_\circ = s_k \cdots s_1 v$ and thus $u = v$. \square

3.3. Small Inversion Sets and Low Elements. In [4], the authors introduced a partial order \preceq on Φ^+ called the *dominance order* defined by:

$$\alpha \preceq \beta \iff (\forall w \in W, \beta \in N(w) \implies \alpha \in N(w)).$$

The ∞ -depth of $\beta \in \Phi^+$ is the number of positive roots strictly dominated by β :

$$\text{dp}_\infty(\beta) := |\{\alpha \in \Phi^+ \mid \alpha \prec \beta\}|.$$

Definition 3.7. Let $n \in \mathbb{N}$, we say that a root $\beta \in \Phi^+$ is *n-small* if $\text{dp}_\infty(\beta) \leq n$ and set $\Sigma_n(W)$ to be the set of *n-small* roots.

A 0-small root is called a *small root* and we write $\Sigma(W) := \Sigma_0(W)$. B. Brink and R. Howlett showed in [4] that Σ_n is finite for $n = 0$. This result was later extended by X. Fu [18] to all $n \geq 0$. The (left) *n-small inversion set* of $w \in W$ is

$$\Sigma_n(w) := N(w) \cap \Sigma_n,$$

and we denote by $\Lambda_n(W) \subseteq \mathcal{P}(\Sigma_n(W))$ the set of all *n-inversion sets*. Since $\Sigma_n(W)$ is finite, $\Lambda_n(W)$ is also finite. We write $\Sigma(w) := \Sigma_0(w)$.

Definition 3.8. An element $w \in W$ is *n-low* if $N(w) = \text{cone}_\Phi(\Sigma_n(w))$. We denote by $L_n(W)$ the set of *n-low* elements in W .

A 0-low element is called a *low element* and we write $L(W) := L_0(W)$. Low elements were introduced by P. Dehornoy, M. Dyer, and the first author in [11], and extended for any $n \in \mathbb{N}$ by M. Dyer and the first author in [13]. We refer the reader to [13, §3.1-§3.3] for more details and examples; examples of low elements are also given in Figure 4. We summarize here some results concerning *n-small* inversion sets and *n-low* elements.

Theorem 3.9 ([13]). *Let $n \in \mathbb{N}$.*

- (a) *The map $\Sigma_n : L_n(W) \rightarrow \Lambda_n(W)$ is injective.*
- (b) *The set $L_n(W)$ of *n-low* elements is finite and closed under join in (W, \leq_R) .*
- (c) *The set of low elements $L(W)$ is a finite Garside shadow in (W, S) .*
- (d) *If (W, S) is finite, affine or with Coxeter graph with edges labelled by $3, \infty$, then the set $L_n(W)$ of *n-low* elements is a finite Garside shadow in (W, S) .*

The statements (a) and (b) are [13, Proposition 3.26], the statement (c) is [13, Theorem 1.1] and (d) is [13, Theorem 1.3 and Theorem 4.17]. We end this discussion by recalling two conjectures from [13]:

- [13, Conjecture 1]: The set $L_n(W)$ is a finite Garside shadow in (W, S) .
- [13, Conjecture 2]: The map $\Sigma_n : L_n(W) \rightarrow \Lambda_n(W)$ is a bijection.

3.4. Low Element Automata and Canonical Automata. Let $n \in \mathbb{N}^*$, the *n-canonical automaton* is the finite automaton $\mathcal{A}_n(W, S)$ over S defined as follows:

- the (finite) set of states is $\Lambda_n(W)$;
- the initial state is $\emptyset (= \Sigma_n(e))$ and all states are final;
- the transitions are: $A \xrightarrow{s} \{\alpha_s\} \cup (s(A) \cap \Sigma_n)$ whenever $\alpha_s \notin A$.

As shown in [13], if $A = \Sigma_n(w)$ then $s \notin D_L(w)$ if and only if $\alpha_s \notin A$, and in this case $\{\alpha_s\} \cup (s(A) \cap \Sigma_n) = \Sigma_n(sw)$. The transitions are thus well defined. Also, one has immediately that if $w = s_1 \cdots s_k$ is reduced, then the path from \emptyset with labels s_1, \dots, s_k ends in the state $\Sigma_n(w)$.

Therefore the *n-canonical automaton* $\mathcal{A}_n(W, S)$ recognizes $\text{Red}(W, S)$, for any $n \in \mathbb{N}$.

The 0-canonical automaton, or simply the *canonical automaton*, was studied by H. Eriksson in his thesis [17] and named in [2, §4.8].

When $L_n(W)$ is a Garside shadow in (W, S) —which we suspect is *always* the case [13, Conjecture 1]—we may consider the associated finite Garside shadow projection and automaton.

Proposition 3.10. *Let $n \in \mathbb{N}$.*

- (a) *The Garside shadow projection $\pi_{L_n(W)}$ is well-defined.*

- (b) The map $\pi_n : \Lambda_n(W) \rightarrow L_n(W)$ defined by $\pi_n(\Sigma_n(w)) := \pi_{L_n(W)}(w)$ is a well-defined surjection.
- (c) The map π_0 induces a totally surjective morphism from the canonical automaton $\mathcal{A}_0(W, S)$ to the automaton $\mathcal{A}_{L(W)}(W, S)$.
- (d) If $L_n(W, S)$ is a Garside shadow in (W, S) , then π_n induces a totally surjective morphism from the n -canonical automaton $\mathcal{A}_n(W, S)$ to the automaton $\mathcal{A}_{L_n(W)}(W, S)$.

Proof. (a) As observed in Remark 2.7, the definition of the Garside shadow projection $\pi_{L_n(W)}$ only requires $L_n(W)$ to contain S and be closed under taking joins, which is guaranteed by Theorem 3.9(b).

- (b) The fact that π_n is surjective follows from the definition of $\Lambda_n(W)$. To prove that π_n is well-defined, let $u, v \in W$ such that $\Sigma_n(u) = \Sigma_n(v)$. We have to show that $\pi_{L_n(W)}(u) = \pi_{L_n(W)}(v)$. By Definition 2.4, it is enough to show that

$$\{g \in L_n(W) \mid g \leq_R u\} = \{g \in L_n(W) \mid g \leq_R v\}.$$

Let $g \in L_n(W)$ such that $g \leq_R u$. By Proposition 2.12, we have $N(g) \subseteq N(u)$, and therefore $\Sigma_n(g) \subseteq \Sigma_n(u) = \Sigma_n(v)$. Since g is n -low we have by definition

$$N(g) = \text{cone}_\Phi(\Sigma_n(g)) \subseteq \text{cone}_\Phi(\Sigma_n(v)) \subseteq N(v).$$

Therefore, again by Proposition 2.12, $g \leq_R v$. This shows the left-to-right inclusion, and we conclude the other inclusion by symmetry.

- (c) This follows from (d) and Theorem 3.9(c).

- (d) We must check the three conditions of Definition 3.1:

- (i) This follows from the fact that $\pi_n(\emptyset) = \pi_n(\Sigma_n(e)) = \pi_{L_n(W)}(e) = e$.
- (ii) This follows since every state in both automata is final and π_n is surjective.
- (iii) By definition of the transitions in $\mathcal{A}_{L_n(W)}(W, S)$ and $\mathcal{A}_n(W, S)$, one must check that if $s \notin D_L(w)$, then $\pi_{L_n(W)}(s\pi_{L_n(W)}(w)) = \pi_{L_n(W)}(sw)$. This follows immediately from Proposition 2.8.

To prove that π_n is totally surjective, it remains to show that, if $w \xrightarrow{s} \pi_{L_n(W)}(sw)$ is a transition in $\mathcal{A}_{L_n(W)}(W, S)$, then $\Sigma_n(w) \xrightarrow{s} \Sigma_n(sw)$ is a transition in $\mathcal{A}_n(W, S)$. But $s \notin D_L(w)$, therefore $\alpha_s \notin \Sigma_n(w)$, and so $\Sigma_n(w) \xrightarrow{s} \Sigma_n(sw)$ is a transition in $\mathcal{A}_n(W, S)$. \square

Remark 3.11. Let $n \in \mathbb{N}$. If the map $\Sigma_n : L_n(W) \rightarrow \Lambda_n(W)$ is a bijection, i.e. [13, Conjecture 2] has a positive answer, then π_n would induce a *isomorphism* from the n -canonical automaton $\mathcal{A}_n(W, S)$ to $\mathcal{A}_{L_n(W)}(W, S)$.

Proposition 3.12. *If W is finite, then the canonical automaton $\mathcal{A}_0(W, S)$ is minimal.*

Proof. Since W is finite, we have $\Phi^+ = \Sigma(W)$. Then in particular $\Sigma(w) = N(w)$ for any $w \in W$ and therefore $L(W) = W = \tilde{S}$, by Proposition 3.6. So $N = \Sigma : L(W) = W = \tilde{S} \rightarrow \Lambda(W)$ is a bijection and therefore $\mathcal{A}_0(W, S)$ and $\mathcal{A}_{\tilde{S}}(W, S)$ are isomorphic. The result follows therefore by Proposition 3.6. \square

The next corollary of Proposition 3.10, together with Corollary 3.5, strengthens the evidence for Conjecture 1.

Corollary 3.13. *The automaton $\mathcal{A}_{\tilde{S}}(W, S)$ associated to the smallest Garside shadow \tilde{S} in (W, S) is a quotient of all canonical automata $\mathcal{A}_n(W, S)$.*

Figure 2 illustrates all of our automata recognizing $\text{Red}(W, S)$, and the maps between them.

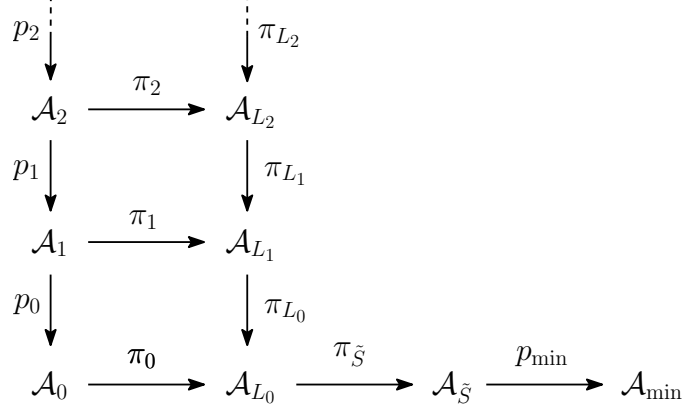


FIGURE 2. Commutative diagram relating the various automata described in this article. The projections π_i are conjectured to be bijections [13, Conjecture 2], and so is p_{\min} (Conjecture 1). The sets $L_i \subseteq W$ are conjectured to be Garside shadows [13, Conjecture 1]; only L_0 is known to be. Finally Conjecture 2 characterizes when the bottom row $p_{\min} \circ \pi_{\tilde{S}} \circ \pi_0$ is an isomorphism.

3.5. Minimality of the canonical automaton. A positive root $\beta \in \Phi^+ = \text{cone}_{\Phi}(\Delta)$ has a unique expression with nonnegative linear combination of vectors in Δ : $\beta = \sum_{s \in S} a_s \alpha_s$, with $a_s \geq 0$; we define the *support* of β to be the set

$$\text{supp}(\beta) := \{s \in S \mid a_s > 0\}.$$

We say that a positive root β is *spherical* if the standard parabolic subgroup $W_{\text{supp}(\beta)}$ is finite, and we write Φ_{sph}^+ for the set of spherical roots.

Spherical roots are always small. Now if the reverse inclusion holds, the following proposition shows that the canonical automaton is minimal, so that one implication in Conjecture 2 is true.

Proposition 3.14. *Let W be irreducible. If $\Sigma = \Phi_{sph}^+$, then $\mathcal{A}_0(W, S)$ is minimal.*

The following proof is inspired by Theorem V.8 in P. Headley's thesis [19].

Proof. Let $\Sigma(u)$ and $\Sigma(v)$ be two equivalent states of $\mathcal{A}_0(W, S)$. This means that for any s_1, s_2, \dots, s_k in S , we have that $s_k \cdots s_1 u$ is reduced if and only if $s_k \cdots s_1 v$ is reduced. We have to show that $\Sigma(u) = \Sigma(v)$.

Assume that they are distinct, so that, up to exchanging the role of u and v , there is $\alpha \in \Sigma(u) \setminus \Sigma(v)$. By assumption, $\Sigma = \Phi_{sph}^+$, so there is $I \subseteq S$ such that W_I is finite and $\alpha \in \Phi_I^+$.

Now we use the decompositions $u = u_I u^I$ and $v = v_I v^I$ in $W_I \times X_I$. The expression wu is reduced if and only if wu_I is reduced, since gu^I is reduced for any $g \in W_I$. So we have that for any $s_1, \dots, s_k \in I$, $s_k \cdots s_1 u_I$ is reduced if and only if $s_k \cdots s_1 v_I$ is reduced.

Since W_I is finite, the automaton $\mathcal{A}_0(W_I, I)$ is minimal by Proposition 3.12. Therefore $\Sigma(u_I) = \Sigma(v_I)$. Note that $\Sigma(u_I) := \Sigma(u) \cap \Phi_I$ and $\Sigma(v_I) := \Sigma(v) \cap \Phi_I$

are small inversion sets for (W_I, I) , by Corollary 2.13 and the definition of small inversion sets. But α was chosen to be in $\Phi_I \cap \Sigma(u)$, which contradicts that $\alpha \in \Sigma(u) \setminus \Sigma(v)$. Therefore, $\Sigma(u) = \Sigma(v)$. \square

We conclude by proving Theorem 1.3.

Proof of Theorem 1.3. (1) This is Proposition 3.12, since $\mathcal{A}_0(W, S)$ and $\mathcal{A}_{\tilde{S}}(W, S)$ are isomorphic in this case.

- (2) Here $\Sigma = \Phi_{sph}^+ = S$; see for instance [11, Proposition 5.1(iii)]. So we are in the case of Proposition 3.14.
- (3) In this case Φ_{sph}^+ consists of the union of all $\Phi_{s,t}^+$ where $m_{st} < \infty$. This is equal to the whole of Σ since the support of a small root is a tree with no ∞ -edge [3]. We conclude again by Proposition 3.14. Note that one can actually give an explicit description of the canonical automaton in this case and prove its minimality directly.
- (4) The fact that the automaton is minimal in this case is due to Eriksson [17, Theorem 80]. Now recall that the Coxeter graph is a simply-laced cycle. Since the support of a small root is a tree [3], we have $\Sigma = \Phi_{sph}^+$ here and the conjecture holds by Proposition 3.14.
- (5) The case of complete graphs was already checked, so one may assume that we have generators s, t, u with $m_{su} = 2$ and $3 \leq m_{st} \leq m_{tu}$. Denote $m = m_{st}$ and $p = m_{tu}$. If $p = \infty$, or if $m = 3$ and $p < 6$, we have $\Sigma = \Phi_{sph}^+$, so Proposition 3.14 gives us the result.

We may now assume $(m = 3 \text{ and } p \geq 6)$ or $(m, p \geq 4)$; in particular W is not finite. Write $c_i = 2 \cos(\pi/i)$. Then $\alpha := us\alpha_t = c_m\alpha_s + \alpha_t + c_p\alpha_u$ is a small root which is not spherical, so that $\Sigma \neq \Phi_{sph}^+$. To show that the conjecture holds we thus need to find two distinct equivalent states in $\mathcal{A}_0(W, S)$.

Now su and tsu are reduced words, with distinct final states in $\mathcal{A}_0(W, S)$ given by $\Sigma(su) = \{\alpha_s, \alpha_u\}$ and $\Sigma(sut) = \{\alpha_s, \alpha_u, \alpha\}$. We have $D_L(su) = D_L(tsu) = \{s, u\}$, so only t can be read from any of these states. Now a quick computation shows $t(\alpha) = \alpha + (c_m^2 + c_p^2 - 1)\alpha_t$. Since $c_m^2 + c_p^2 - 1 \geq 2$ for all considered values of m and p , we have $t(\alpha) \notin \Sigma$ and therefore $\Sigma(tsu) = \Sigma(tsut)$. This shows that $\Sigma(su)$ and $\Sigma(tsu)$ are equivalent states, and thus that $\mathcal{A}_0(W, S)$ is not minimal. \square

Remark 3.15 (On Conjectures 1 and 2). Using Sage [23, S⁺09], we wrote code to compute the set of small roots. We used these to compute the canonical automaton, from which we determined the minimal automaton. It is simple to test if a given small roots is spherical by examining the simple roots that occur in its support, from which we are able to check Conjecture 2. This code is sufficiently fast to compute examples in rank 5—for example, we determined that the minimal automaton for \tilde{D}_5 has size 58965.

We also wrote a naive implementation to determine the minimal Garside shadow using Definition 2.2 to check Conjecture 1. This code finishes in a few minutes on standard hardware in rank four (and below), but already takes longer than several hours in rank five.

Our software confirms that Conjectures 1 and 2 hold for all Coxeter groups of rank 4 with edge labels less than 10. Figure 3 includes data for a few selected Coxeter groups of low rank.

Name	Coxeter Diagram	$ A_0(W, S) $	$ A_{\tilde{S}}(W, S) $	$ A_{min}(W, S) $	$ \Sigma $	$ \Phi_{sph}^+ $
\tilde{A}_2		16	16	16	6	6
\tilde{C}_2		25	24	24	8	7
\tilde{G}_2		49	41	41	12	8
\tilde{A}_3		125	125	125	12	12
\tilde{C}_3		343	317	317	18	15
\tilde{B}_3		343	315	315	18	15
		92	92	92	15	15
		164	164	164	21	21
		91	80	80	18	14
		100	90	90	30	25

FIGURE 3. Numerical data for selected Coxeter groups. Note that in affine type \tilde{W}_n , $|A_0(W, S)| = (h + 1)^n$ and $|\Sigma| = nh$, where h is the Coxeter number of the corresponding finite Weyl group.

3.6. Canonical automata and Shi arrangements. We end this article by describing some rank 3 examples of automata. It turns out these examples can be drawn in a very nice way: their states form a convex set in the (dual of the) geometric representation of (W, S) . The reason in the affine case is related to a property of the Shi arrangement, which leads us to discuss a generalization of the Shi arrangement for any Coxeter system.

Let Φ_0 be a reduced, irreducible, crystallographic root system of rank r for a finite Weyl group W_0 in a real vector space V_0 with W_0 -invariant positive definite scalar product $\langle \cdot, \cdot \rangle$. Let Φ_0^+ be a choice of positive roots, let $\Delta_0 = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

be the corresponding simple roots. The *height* of a positive root $\alpha = \sum_{i=0}^n c_i \alpha_i$ is $\sum_{i=0}^n c_i$; for example, the *highest root* α_h in Φ_0^+ has height $h - 1$, where h is the Coxeter number of W_0 . Define $V := V_0 \oplus \mathbb{R}\delta$ and define the set of affine roots to be

$$\Phi := \{\alpha + k\delta \mid \alpha \in \Phi \text{ and } k \in \mathbb{Z}\}.$$

The positive affine roots are $\Phi^+ := \{\alpha + k\delta \mid \alpha \in \Phi_0^+ \text{ and } k \geq 0 \in \mathbb{Z}\} \cup \{\alpha + k\delta \mid \alpha \in -\Phi_0^+ \text{ and } k > 0 \in \mathbb{Z}\}$, and the simple affine roots are $\Delta := \Delta_0 \cup \{\alpha_0\}$, where $\alpha_0 := -\alpha_h + \delta$.

For $\alpha \in \Phi_0$ and $k \in \mathbb{Z}$, we consider in V_0 , seen as an affine space, the affine hyperplane

$$H_{\alpha,k} := \{x \in V_0 \mid \langle x, \alpha \rangle = k\}.$$

The affine Weyl group W is the group generated by affine reflections in the simple affine hyperplanes: $H_{\alpha,0}$ for $\alpha \in \Delta_0$ and $H_{\alpha_h,1}$. The fundamental alcove \mathcal{K} is the (interior of the) compact region bounded by the simple affine hyperplanes. The closure of \mathcal{K} is a fundamental domain for the action of W .

The *n-Shi arrangement* is the collection of hyperplanes

$$\text{Shi}_n(W) := \{H_{\alpha,k} \mid \alpha \in \Phi^+, -n+1 \leq k \leq n\}.$$

We abbreviate $\text{Shi}(W) := \text{Shi}_1(W)$, and call it the *Shi arrangement*². The roots corresponding to the hyperplanes in $\text{Shi}_n(W)$ coincide with the n -small roots, so that Σ_n can be thought of as a generalization of the n -Shi arrangement to *any* Coxeter group. The crystallographic affine root system Φ is easily and bijectively convertible to a root system; so the dominance order and n -small roots are well-defined in a crystallographic root system. In particular, the only relations in the dominance order on Φ^+ are $\alpha + k\delta \preceq \alpha + \ell\delta$ for $\alpha \in \Phi, k \leq \ell \in \mathbb{Z}$; see [13, Example 3.9]. We obtain therefore the following proposition.

Proposition 3.16. *If (W, S) is of affine type, then*

$$\text{Shi}_n(W) = \{H_\alpha \mid \alpha \in \Sigma_n(W)\}.$$

The Shi arrangements for types \tilde{A}_2 and \tilde{C}_2 are drawn in Figure 4.

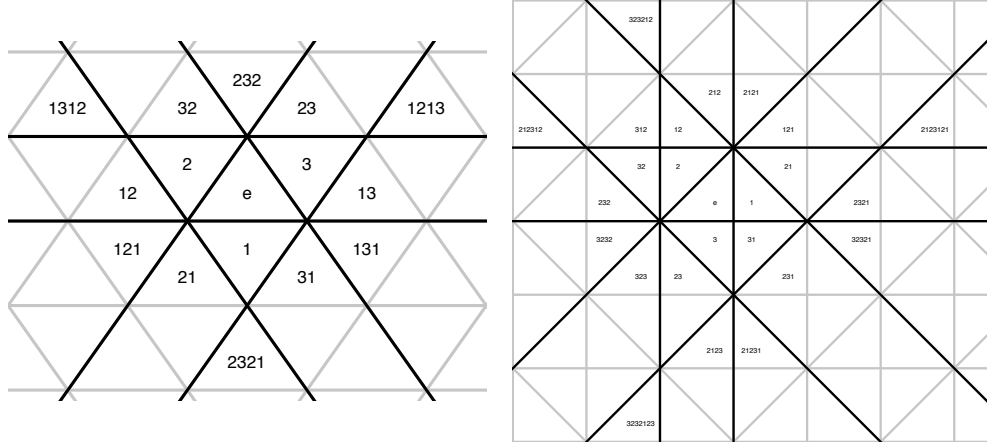
In affine type, the small inversion sets have previously been studied under the guise of the minimal alcoves of the n -Shi arrangement. More precisely, for W of affine type, $\Lambda_n(W) = \{w \in W \mid \{w(\alpha_s) \mid s \in D_L(w)\} \subseteq \Sigma_n\}$. The corresponding statement for n -low elements and general type is given as [13, Conjecture 2], restated above in §3.

It turns out that there are $(nh + 1)^r$ n -low elements in affine type. The reason for this is that the inverses of such elements coalesce into an $(nh + 1)$ -fold dilation of the fundamental alcove.

Theorem 3.17 (J. Y. Shi). *Let \mathcal{K} be the fundamental alcove for an affine Weyl group W , and let h be the Coxeter number of the corresponding finite Weyl group W_0 . Then*

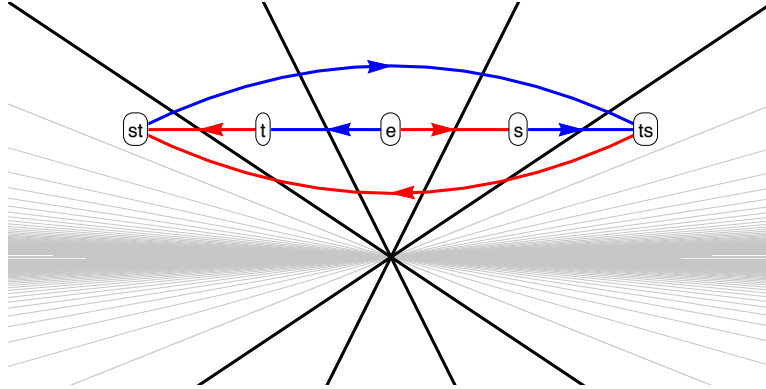
$$\{w^{-1}\mathcal{K} \mid w \text{ an } n\text{-low element}\} \cong (nh + 1)\mathcal{K}.$$

²This was called the *sandwich arrangement* in [19]


 FIGURE 4. A_2 and C_2 Shi arrangements.

In particular, the alcoves corresponding to the inverses of n -low elements form a convex set. This theorem is illustrated for the infinite dihedral group $\tilde{A}_1 = I_2(\infty)$ in Figures 1 and 5, which show the automata built from the 1- and 2-low elements, respectively. Figure 6 illustrates this theorem for types \tilde{A}_2 and \tilde{C}_2 , simultaneously drawing the automaton.

We note that convexity does not necessarily hold for the subset of alcoves coming from the inverses of elements in \tilde{S} , as seen for example in Figure 6—for \tilde{C}_2 , $\tilde{S} = \Sigma \setminus \{s_1 s_3 s_2\}$.


 FIGURE 5. The automaton $\mathcal{A}_1(I_2(\infty), S)$, drawn using Theorem 3.17.

On the basis of the affine rank three examples, it is tempting to conjecture that equivalent states are given by intersecting *intervals* in the weak order with $L_n(W)$. The (non-affine) triangle group $(5, 3, 5)$ is a counterexample to this claim.

When the Coxeter system (W, S) is of *indefinite type*, i.e., W is not finite nor affine, the *isotropic cone* $Q = \{x \in V \mid B(x, x) = 0\}$, and the region where $x \in V$ verifies $Q(x, x) < 0$, are nonempty. In this case, following [21, 14], we consider the *projective representation* for (W, S) associated to the geometric representation of

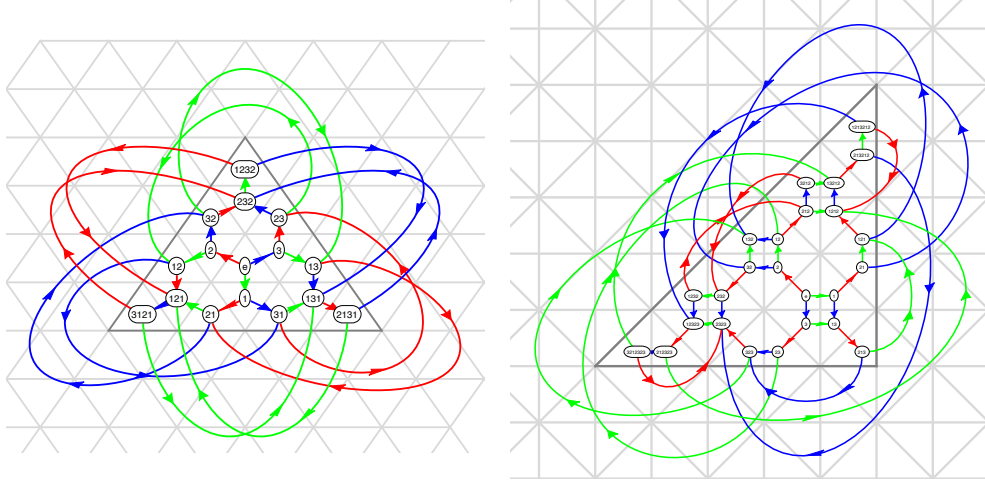


FIGURE 6. The automata $\mathcal{A}_0(\tilde{A}_2, S)$ and $\mathcal{A}_0(\tilde{C}_2, S)$, drawn using Theorem 3.17. A [green, red, blue] edge represents multiplication by $[s_1, s_2, s_3]$. There is one omitted (red) edge between 132 and 213 in $\mathcal{A}_0(\tilde{C}_2, S)$.

(W, S) , with roots system Φ and simple system Δ in §2.4. More precisely, since $\Phi = \Phi^+ \sqcup \Phi^-$ is encoded by the set of positive roots Φ^+ , we represent Φ by an ‘affine cut’ $\hat{\Phi}$: there is an affine hyperplane V_1 in V *transverse to* Φ^+ , i.e., for any $\beta \in \Phi^+$, the ray $\mathbb{R}^+\beta$ intersects V_1 in a unique nonzero point $\hat{\beta}$. So $\mathbb{R}\beta \cap V_1 = \{\hat{\beta}\}$ for any $\beta \in \Phi$. The *set of normalized roots* $\hat{\Phi} = \{\hat{\beta} \mid \beta \in \Phi\}$ is contained in the compact set $\text{conv}(\hat{\Delta})$ and therefore admits a set E of accumulation points called *the set of limit roots*, which verifies $E \in \hat{Q}$. The group W acts on $\hat{\Phi} \sqcup E \cup \text{conv}(E)$ componentwise: $w \cdot x = \widehat{w(x)}$.

Now, the role of the affine space V_0 for an affine Coxeter system (W, S) with the tiling obtained by the action of W on the fundamental alcove \mathcal{K} is replaced for indefinite Coxeter systems by a tiling of the *imaginary convex body* $\text{conv}(E)$ by the projective action of W on the non-empty fundamental region

$$K = \{x \in \text{conv}(\hat{\Delta}) \mid B(x, \alpha_s) \geq 0, \forall s \in S\}.$$

Denote $H_\alpha = \{x \in V \mid B(x, \alpha) = 0\}$, then K is the region of $\text{conv}(\hat{\Delta})$ bounded by the hyperplanes H_α . This is illustrated in Figure 7 and Figure 8; see also [14, Figures 2 and 14]. We refer the reader to [14] for more details.

In view of Proposition 3.16, it is natural to give the following definition.

Definition 3.18. Let (W, S) be an indefinite Coxeter system. The *n-Shi arrangement* of (W, S) is the collection of hyperplanes

$$\text{Shi}_n(W, S) := \{H_\alpha = \{x \in V \mid B(x, \alpha) = 0\} \mid \alpha \in \Sigma_n(W)\}.$$

If [13, Conjecture 2], restated above in §3, is true, it would mean that the set $L_n(W)$ of n -low elements parameterized the region of the n -Shi arrangement. Furthermore, each region $\text{Shi}_n(W, S)$ would have a unique minimal-length region of the form $w \cdot K$ with $w \in L_n(W)$. Moreover, we observed in numerous cases in rank 3 and 4 the following statement.

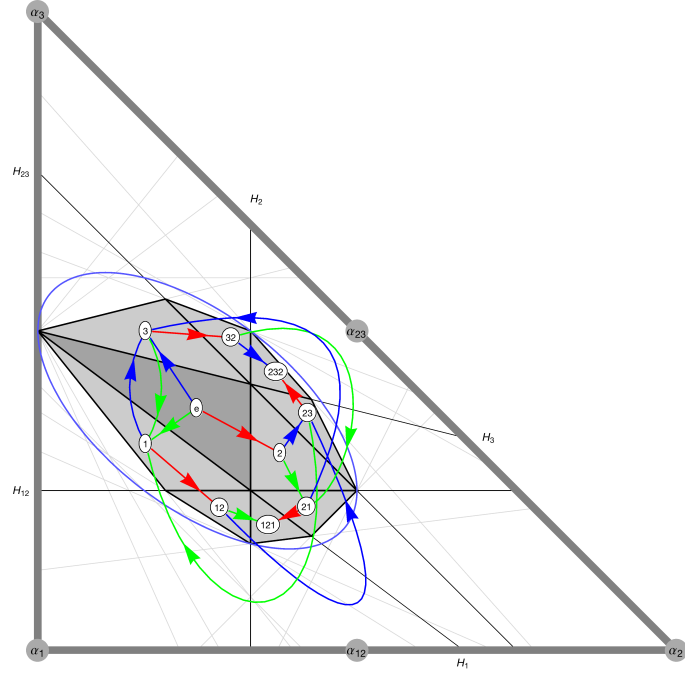


FIGURE 7. The automaton $\mathcal{A}_0(W, S)$ for W the triangle group $(3, 3, \infty)$.

Conjecture 3. *Let $n \in \mathbb{N}$, then the subset $\bigcup_{w \in L_n(W)} w^{-1} \cdot K$ of the imaginary convex body is convex.*

Reasonable evidence for Conjecture 3 is supplied by the fact that $L_n(W)$ is closed under taking suffixes. Figure 8 illustrates Conjecture 3 for several rank three examples.

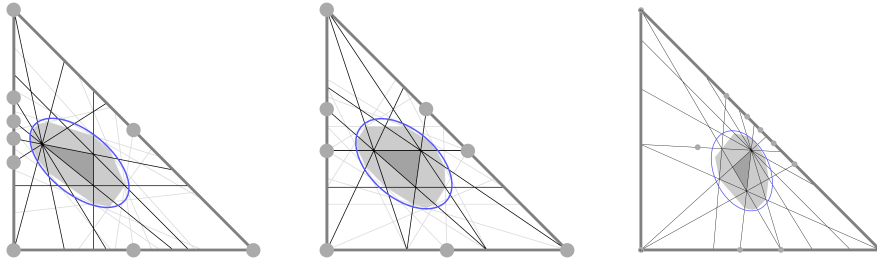


FIGURE 8. The regions $\bigcup_{w \in L_n(W)} w^{-1} \cdot K$ for the triangle groups $(3, 3, 6)$, $(3, 4, 4)$, and $(4, 7, 2)$.

Acknowledgments. This work was initiated in LaCIM (Montreal) while Philippe Nadeau was visiting thanks to a research travel grant from the *Laboratoire International Franco-Québécois de Recherche en Combinatoire (LIRCO)*. The first author (CH) thanks Christophe Reutenauer for interesting conversations regarding minimality of automata and restriction to standard parabolic subgroups. We thank an anonymous referee for suggesting that we provide more evidence for Conjectures 1 and 2.

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(Christophe Hohlweg) UNIVERSITÉ DU QUÉBEC À MONTRÉAL, LACIM ET DÉPARTEMENT DE MATHÉMATIQUES, CP 8888 SUCC. CENTRE-VILLE, MONTRÉAL, QUÉBEC, H3C 3P8, CANADA
E-mail address: hohlweg.christophe@uqam.ca
URL: <http://hohlweg.math.uqam.ca>

(Philippe Nadeau) CNRS & INSTITUT CAMILLE JORDAN, UNIVERSITÉ CLAUDE BERNARD LYON 1, 69622 VILLEURBANNE CEDEX, FRANCE
E-mail address: nadeau@math.univ-lyon1.fr
URL: <http://math.univ-lyon1.fr/~nadeau>

(Nathan Williams) UNIVERSITÉ DU QUÉBEC À MONTRÉAL, LACIM, CP 8888 SUCC. CENTRE-VILLE, MONTRÉAL, QUÉBEC, H3C 3P8, CANADA
E-mail address: nathan.f.williams@gmail.com
URL: <http://thales.math.uqam.ca/~nwilliams/>